Simulation of electromagnetic wave propagation inside and outside a rectangular waveguide

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Abstract

The problem consists of an electromagnetic wave incident on one of the faces of a rectangular waveguide.Part of the wave is coupled to the waveguide and part of it propagates outside,assumed to be free space.The shape of the propagating wavefront is to be found out,both inside and outside the waveguide. The electric field propagation can be found out by solving Maxwell's equations both inside and outside the waveguide, with proper boundary conditions. The numerical technique used here to solve Maxwell's equations is Finite Difference Time Domain method,which utilizes central difference approximation. Two different boundary conditions are imposed on the waveguide and on the free space boundary.Waveguide walls are characterised by PEC and free space boundary is characterised by PML boundary conditions.The code is written in 'C' language,output is plotted in MATLAB.An mpeg movie is created from the code generated data, showing the evolution of the electric field with time inside the waveguide

MATHEMATICAL FORMULATION

The mathematical formulation of the problem consists of the specification of the waveguide parameters,imposing the proper boundary conditions, applying central difference approximation to the Maxwell's equations and discreetizing them in all the 3 co-ordinates,which would be used in the program.

waveguide specification

The dimension of the waveguid is a=2.29 cm,b=1.02cm.The frequency of the incident wave is 10 Ghz. The waveguide length is $5 \times \lambda$,where λ is the wavelength of the incident wave which in this case is 3 cm.

Boundary Conditions

The problem spans in two mediums,inside the waveguide and outside i.e in free space.The total space including these regions are discreetized into grids or cells,with different boundary conditions for both the regions.The number of cells per wavelength is 25.The boundary surface of the waveguide is a perfect electrical conductor.PEC boundary condition is

1. Tangential component of the electric field is 0 in the waveguide at the boundary.
2. En is not necessarily zero in the wave guide at the boundary as there may be surface charges on the conducting walls

The continuity of the magnetic field implies that normal component of magnetic field is 0 at the boundary.There are currents induced in the guides but for perfect conductors these can be only surface currents. Hence, there is no continuity for H_t.

Far field conditions

The far field conditions must be satisfied at the end of the vaccum grid.Here perfectly matched layers are imposed as boundary conditions.These represent the absorbing boundary

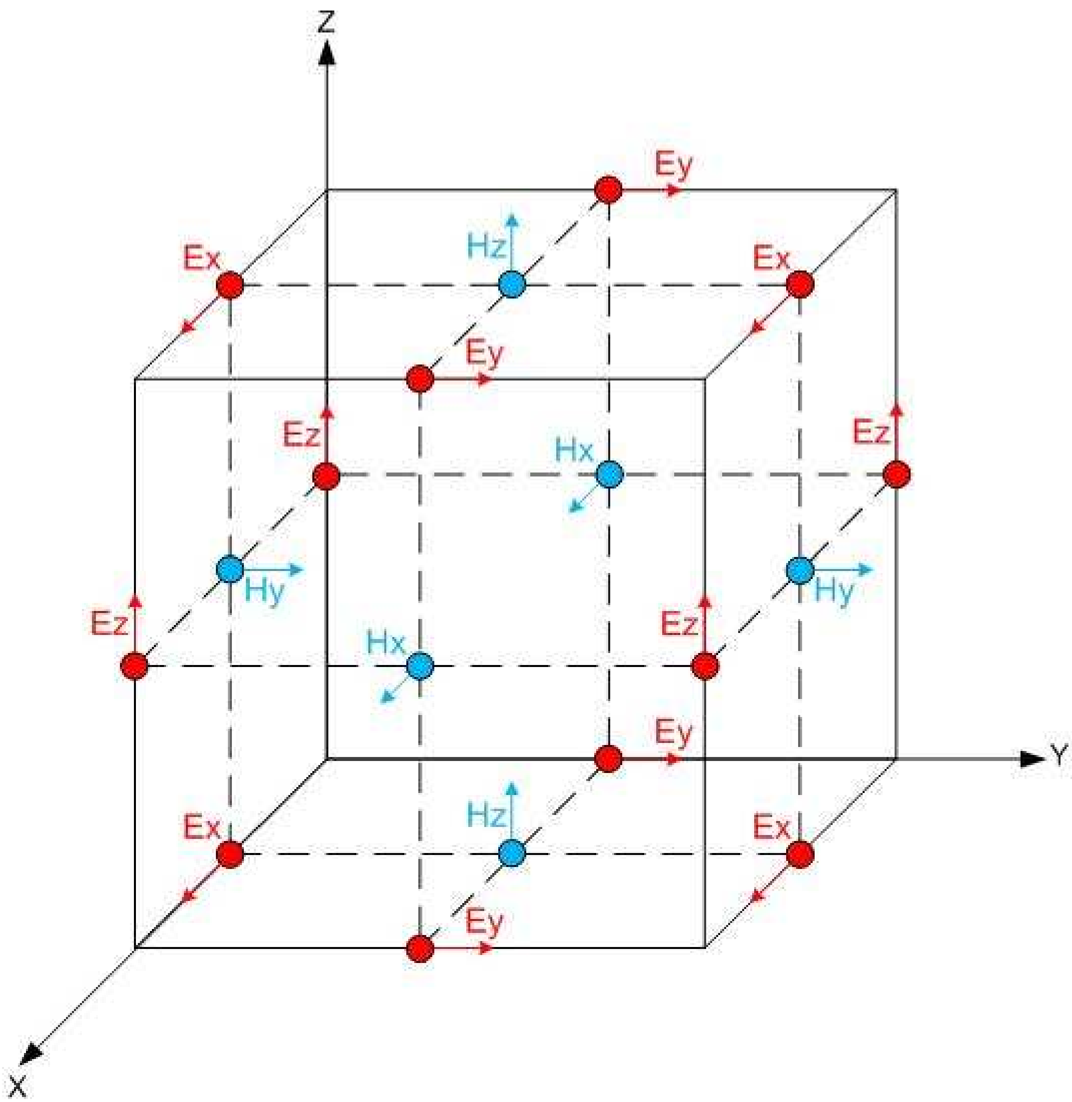

FIG. 1. one single fdtd cell geometry

condition that the field must vanish at infinity.But numericaly no grid can extend upto infinity.The last few surfaces of the vaccum cell are imposed with PML conditions.A PML is designed to absorb waves coming to its direction,Wherever an x derivative $\partial/\partial x$ appears in the wave equation, it is replaced by:

$$\frac{\partial}{\partial x} \rightarrow \frac{1}{1+\frac{i\sigma(x)}{\omega}} \frac{\partial}{\partial x} \tag{1}$$

Where ω is theangular frequency and σ is some function of x.Wherever σ is positive,propagating waves are attenuated,because

$$exp[i(kx-\omega t)] \rightarrow exp[i(kx-\omega t) - \frac{k}{\omega}\int \sigma(x')dx'] \tag{2}$$

This transformation of coordinates causes waves to attenuate whenever they have a dependance of $exp(ikx)$.

source implementation

The length of the guide is along x direction, the width along y direction and the height along z direction.Plane wave emanates from the $x = 0$ face. Thus all the E_z components on the $x = 0$ face are taken as the stimulus. The source is taken as a gaussian pulse expressed as

$$E_z(i, j, k)(t) = Aexp[-c(t - t_0/t_\omega)^2] \tag{3}$$

Where c is a constant.

Specification of cells

The number of cells on each direction i.ex,y,z are taken as the same,here it is 50.The first 25 cells on each direction corressponds to the inside of the waveguide and the outermost of these cells on each direction provide the PEC boundary.The next 20 cells represent the free space grid and the last 5 cells on each direction represents the perfectly matched layers, where the wave gradually decays without any reflection.Say, n_i is the total number of cells in each direction.(i=x, y, z), then the space differntial i.e di is given by

$$dy = guidewidth/n_y \tag{4}$$

$$dz = guideheight/n_z \tag{5}$$

dx is chosen to be the smallest of either dy or dz. The time step is defined as

$$dt = 1/(c \times sqrt[(1/dx^2) + (1/dy^2) + (1/dz^2)]) \tag{6}$$

Total stimulated time is given by

$$T = maxiteration \times dt \tag{7}$$

memory allocation

Mesh is set up such that the tangential E vectors form the outer faces of the simulation volume. Say there are n_x,n_y, n_z cells in the mesh, but the number of E_x vector components are given by

$$n_x(n_y+1)(n_z+1) \tag{8}$$

The number of E_y components are given by

$$(n_x+1)n_y(n_z+1) \tag{9}$$

similarly for E_z components also.
The H arrays are staggerred half a step off from the E arrays in every direction,thus H arrays are one cell smaller than E cells in each direction.
Thus outer faces of the mesh consist of E componnts only and the inner mesh consists of the H components.

Update equations

For a source free medium(vaccum) the maxwell's equations are

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0} \bigtriangledown \times H \tag{10}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \bigtriangledown \times E \tag{11}$$

Where E and H are vectors in 3 Dimensions. These equations interms of the x,y,z components can be written as

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_0}[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}] \tag{12}$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon_0}[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}] \tag{13}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon_0}[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}] \tag{14}$$

The equations for the H vector interms of its components are

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu_0}[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}] \tag{15}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}] \tag{16}$$

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0}[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}] \tag{17}$$

Equations5to10can be interpreted as a small change in $E_x$over time is related to a change in$H_y$and$H_z$in space. Finite Difference approximation can be used for the above equations.The resulting equations can be expressed as

$$\begin{aligned}\frac{E_x(z', t' + \frac{\triangle t}{2}) - E_x(z', t' - \frac{\triangle t}{2})}{\triangle t} = &\frac{1}{\epsilon_0}[\frac{H_z(y' + \frac{\triangle y}{2}, t') - H_z(y' - \frac{\triangle y}{2}, t')}{\triangle y} \\ &- \frac{H_y(z' + \frac{\triangle z}{2}, t') - H_y(z' - \frac{\triangle z}{2}, t')}{\triangle z}]\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{E_y(z', t' + \frac{\triangle t}{2}) - E_y(z', t' - \frac{\triangle t}{2})}{\triangle t} = &\frac{1}{\epsilon_0}[\frac{H_x(z' + \frac{\triangle z}{2}, t') - H_x(z' - \frac{\triangle z}{2}, t')}{\triangle z} \\ &- \frac{H_z(x' + \frac{\triangle x}{2}, t') - H_z(x' - \frac{\triangle x}{2}, t')}{\triangle x}]\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{E_z(z', t' + \frac{\triangle t}{2}) - E_z(z', t' - \frac{\triangle t}{2})}{\triangle t} = &\frac{1}{\epsilon_0}[\frac{H_y(x' + \frac{\triangle x}{2}, t') - H_y(x' - \frac{\triangle x}{2}, t')}{\triangle x} \\ &- \frac{H_x(y' + \frac{\triangle z}{2}, t') - H_x(y' - \frac{\triangle z}{2}, t')}{\triangle y}]\end{aligned} \tag{20}$$

Similar equations can be written for H field with central difference approximation. Alittle manipulation gives the update equations used in the code.These are

$$\begin{aligned}E_x(z', t' + \frac{\triangle t}{2}) = E_x(z', t' - \frac{\triangle t}{2}) + &\frac{\triangle t}{\epsilon_0}[\frac{H_x(z' + \frac{\triangle z}{2}, t') - H_x(z' - \frac{\triangle z}{2}, t')}{\triangle z} \\ &- \frac{H_z(x' + \frac{\triangle x}{2}, t') - H_z(x' - \frac{\triangle x}{2}, t')}{\triangle x}]\end{aligned} \tag{21}$$

$$\begin{aligned}E_y(z', t' + \frac{\triangle t}{2}) = E_y(z', t' - \frac{\triangle t}{2}) + &\frac{\triangle t}{\epsilon_0}\frac{H_x(z' + \frac{\triangle z}{2}, t') - H_x(z' - \frac{\triangle z}{2}, t')}{\triangle z} \\ &- \frac{H_z(x' + \frac{\triangle x}{2}, t') - H_z(x' - \frac{\triangle x}{2}, t')}{\triangle x}]\end{aligned} \tag{22}$$

$$\begin{aligned}E_z(z', t' + \frac{\triangle t}{2}) = E_z(z', t' - \frac{\triangle t}{2}) + &\frac{\triangle t}{\epsilon_0}\frac{H_y(x' + \frac{\triangle x}{2}, t') - H_y(x' - \frac{\triangle x}{2}, t')}{\triangle x} \\ &- \frac{H_x(y' + \frac{\triangle z}{2}, t') - H_x(y' - \frac{\triangle z}{2}, t')}{\triangle y}]\end{aligned} \tag{23}$$

These equations are directly used to generate volumetric data for the simulation of the electromagnetic wave.

LOGIC OF THE PROGRAM

The logic is defined as:

1. Define the parameters of the model waveguide.Discreetize the total space into grids.Specify the number of cells in each direction.Define the space step.

2. Define the time step.Allocate dynamic memory for the field values to be calculated in each iteration.

3. Set the boundary conditions and the source.

4. For each time step,using the update equations for both the E and H fields,the propagation of the electromagnetic field is calculated through the whole grid. this process is repeated for each time step.

5. All the data for this time and space stepping is outputed in .txt files and each file,when plotted,gives a picture of the E field propagation. When all these files are poltted sequentialy,a picture of the wavefront propagating through the waveguide is visualised.

Complexity of the program

The program uses the update equations as the basic equations for calculations.Equations 21 to 23 are the update equations.During each of the time step,the field values are calculated $n_x \times n_y \times n_z$, n_x,n_y,n_z are the number of cells in the x, y, z directions respectively.The output of the program gives the E_z value only.For this, the complexity of the algorithm is of the order n^3 where it is assumed that n_x=n_y=n_z.The complexity for all time steps is $n^3 \times T$ where T is the total number of timesteps.

Error propagation

The error in this method comes from the fact that the total no of cells multiplied by the space steps must equal to the dimensions of the waveguide. In this case $n_y \times dy =$ Guide width in meters,and $n_z \times dz =$ guide height in meters. dx is chosen to be the smallest of dy,dz.Another source of error is the size of the time step.This time step size should be such

that the plane wave travels a single cell length in a single time step.Otherwise FDTD can not keep up with signal propagation, because FDTD computes a cell only from its immediate neighbours.

DATA AND ANALYSIS OF THE ACTUAL PROGRAM PERFORMANCE

The program output was interms of 200 .txt files.These files contain 3D data for the E_z field vector in each timestep. The performance of the program was listed in another .txt files whose contents are displayed below.

program performance

Dynamically allocated bytes : 1173832 bytes.

Data and figures

The program uses one thousand time steps.In each time step,the program calculates the E_z values throughout the grid.Two hundred data files are generated to simulate the wave propagation through the waveguide.These files are plotted using MATLAB environment.An mpeg movie has also been created using the data files.

Fig(2)

Fig(2) is a plot of the simulated time against the number of space steps.Number of time steps is plotted along x axis and the simulation time is plotted along the y axis.The relationship is linear as shown in the graph.As the wave propagates inside the waveguide,the complexity increases and the system time also increases.Fig(2) supports this observation.

Fig(3)and Fig(4)

Fig(3) is a three dimensional plot of E_z at a particular time step.The plot is generated using the data file for the 995 th time step.The software used for plotting was MATLAB. Fig(4) is a three dimensional plot of the propagating wavefront.

h

TABLE I.

Below are the data for system time taken to compute the E_z values in the grid in each time step.The space steps calculated during the program are $dx = 0.00113333$ meters,$dy = 0.001145$ meter,$dz = 0.00113333$.The total simulation region is $0.0229 \times 0.0102 \times 0.149896$ $meter^3$.The total time for simulation is 2.19002×10^{-09} seconds.The complete data is not mentioned here.

time step	system time
1	2.19002e-12sec
2	4.38005e-12sec
3	6.57007e-12sec
4	8.7601e-12sec
5	1.09501e-11sec
6	1.31401e-11sec
7	1.53302e-11sec
8	1.75202e-11sec
9	1.97102e-11sec
422	9.2419e-10
423	9.2638e-10
424	9.2857e-10
425	9.3076e-10
426	9.3295e-10
427	9.3514e-10
428	9.3733e-10
591	1.2943e-09
592	1.29649e-09
593	1.29868e-09
594	1.30087e-09
595	1.30306e-09
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997	2.18345e-09
998	2.18564e-09
999	2.18783e-09

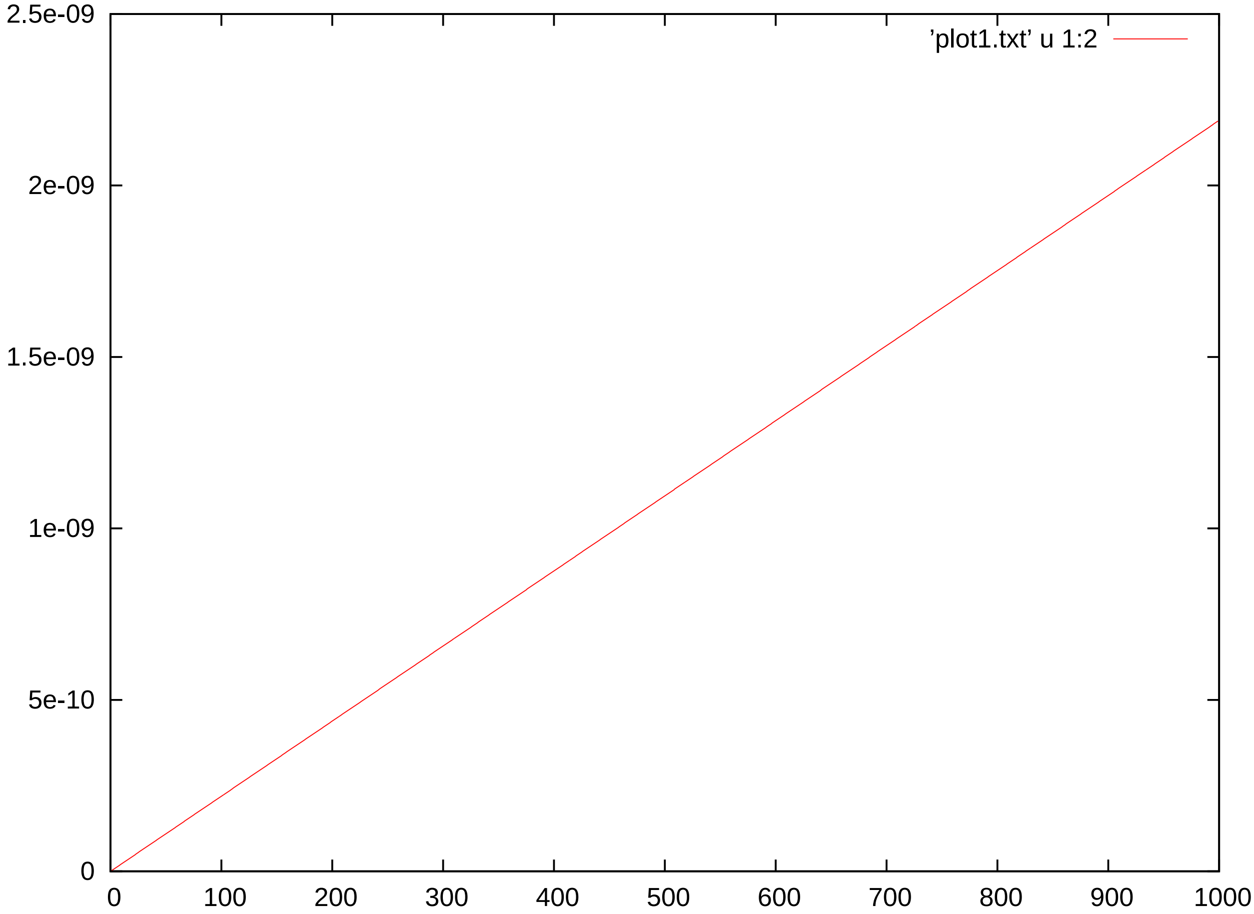

FIG. 2. simulation time vs time step number

A hollow rectangular waveguide can propagate TM and TE modes but not TEM modes.The TE modes are characterised by fields with E_z=0 while H_z must satisfy the wave equation

$$(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 x} + k_c^2)h_z(x, y) = 0 \tag{24}$$

Similarly the E_z fields in the TM modes satisfy the wave equation

$$(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 x} + k_c^2)e_z(x, y) = 0 \tag{25}$$

Where

$$E_z(x, y, z) = e_z(x, y)exp(-j\beta z) \tag{26}$$

and

$$H_z(x, y, z) = h_z(x, y)exp(-j\beta z) \tag{27}$$

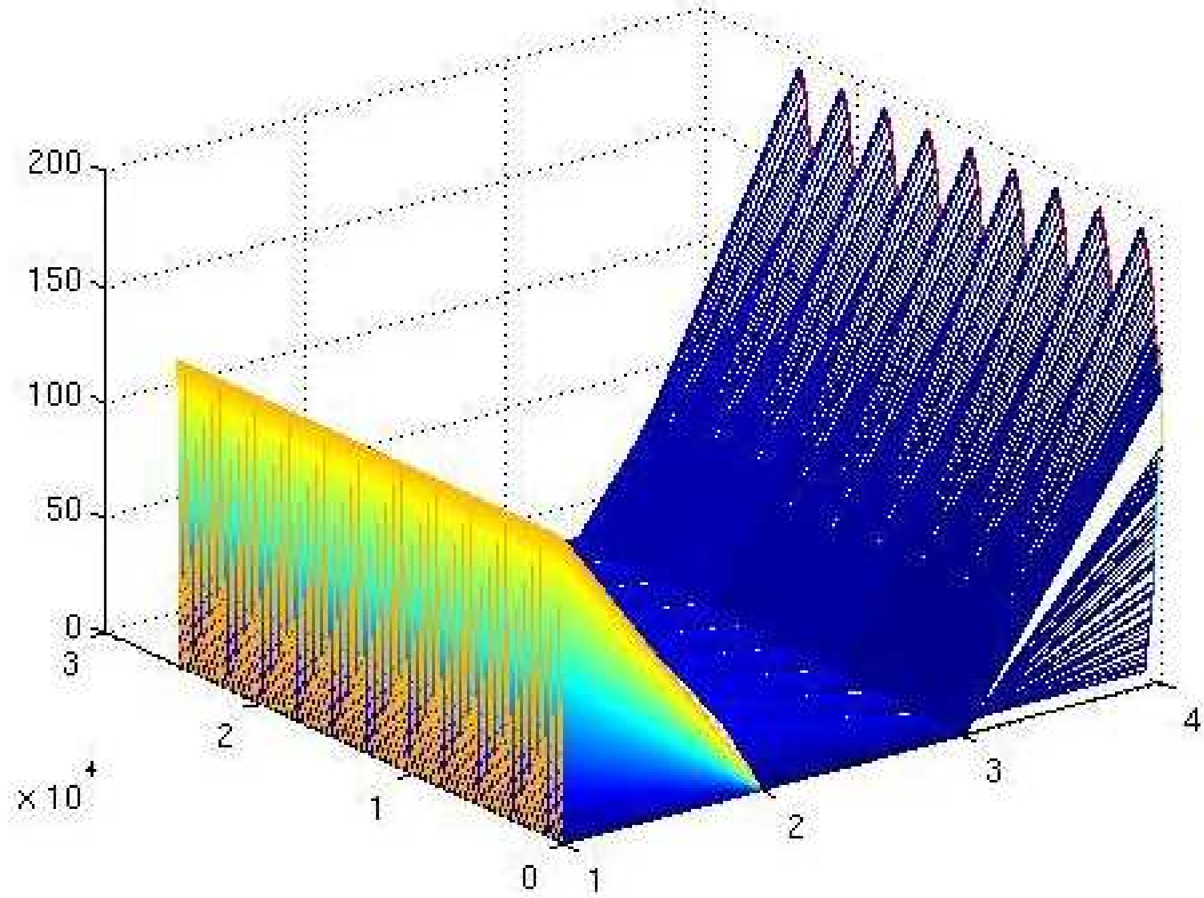

FIG. 3. shape of the wavefront in one time step

$$k_c^2 = k^2 - \beta^2 \tag{28}$$

k_c is the cut off wave number.The general solution can be found out for E_z by using separation of variables.We use

$$e_z(x, y) = X(x)Y(y) \tag{29}$$

The general solution for this equation is

$$e_z(x, y) = (Acosk_x x + Bsink_x x)(Ccosk_y y + Dsink_y y) \tag{30}$$

The boundary conditions are directly applied to the e_z component which are

$$e_z(x, y) = 0 \tag{31}$$

at $x = 0, a$.

$$e_z(x, y) = 0 \tag{32}$$

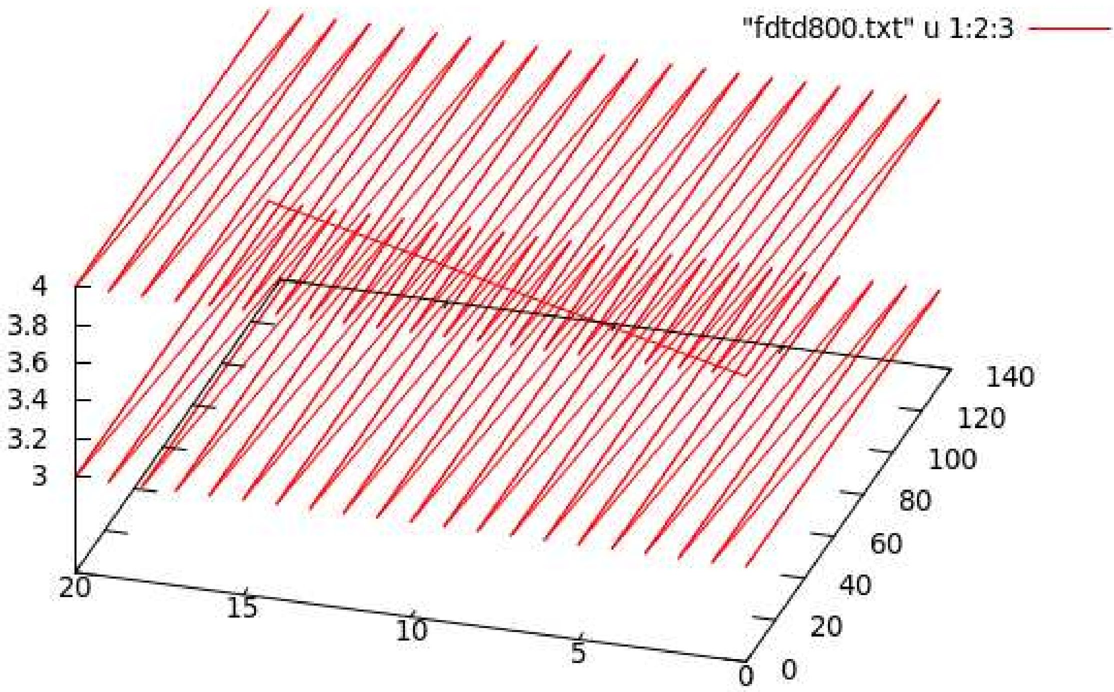

FIG. 4. Propagation of the wavefront

at$y = 0, b$.

Applying 31 and 32 we can get the solution that $A = 0$ and $k = m\pi/a$ for $m = 0, 1, 2, 3....$ Also we get

$C = 0$ and $k_y = n\pi/b$ for $n = 0, 1, 2, 3....$

The solution for E_z can thus be written as

$$E_z(x, y, z) = B_mnsin\frac{m\pi x}{a}sin\frac{n\pi y}{b}exp(-j\beta z) \tag{33}$$

The transverse field components for the TM_mn mode can be written as

$$E_x = frac{-}j\beta m\pi ak_c^2B_mncos\frac{m\pi x}{a}sin\frac{n\pi y}{b}exp(-j\beta z) \tag{34}$$

$$E_y = frac{-}j\beta n\pi bk_c^2 B_m n sin\frac{m\pi x}{a}cos\frac{n\pi y}{b}exp(-j\beta z) \tag{35}$$

$$H_x = fracj\omega\epsilon n\pi bk_c^2 B_m n sin\frac{m\pi x}{a}cos\frac{n\pi y}{b}exp(-j\beta z) \tag{36}$$

$$H_y = \frac{-j\omega\epsilon m\pi}{ak_c^2}B_m n cos\frac{m\pi x}{a}sin\frac{n\pi y}{b}exp(-j\beta z) \tag{37}$$

From equations 34 to 36 we can say that these expressions are identicaly zero if either of m or n is zero.So the lowest order TM mode possible is $TM_1 1$. Its cut off frequency is

$$f_c 11 = \frac{1}{2\pi sqrt(\mu\epsilon)}sqrt((\frac{\pi}{a})^2 + (\frac{\pi}{b})^2)) \tag{38}$$

For this problem $a = 2.29cm$ and $b = 1.02cm$.Thus the cutoff frequency for $TM_1 1$ mode is 16.09 GHz.$TM_1 1$ mode is calculated because the program outputs only E_z values which are possible for $TM_1 1$ modes.

TEST FOR THE CORRECTNESS OF THE SOLUTION

Figure 5 shows the electric field distribution in the waveguide for the $TM_1 1$ mode.Fig(4) showed a similar nature of propagation of the electric field vectors. From that it can be said that the program was able simulate the propagation of one of the eigenmodes, i.e the $TM_1 1$ mode.

Although the free space was not simulated in this program,so the propagation of the wave outside the guide was not found out, so there is further room for improvement in this program.

REFERENCE

1. The waveguide specificationwas taken from David K. Cheng, Field and Wave Electromagnetics, 2nd ed.,pages 554-555.

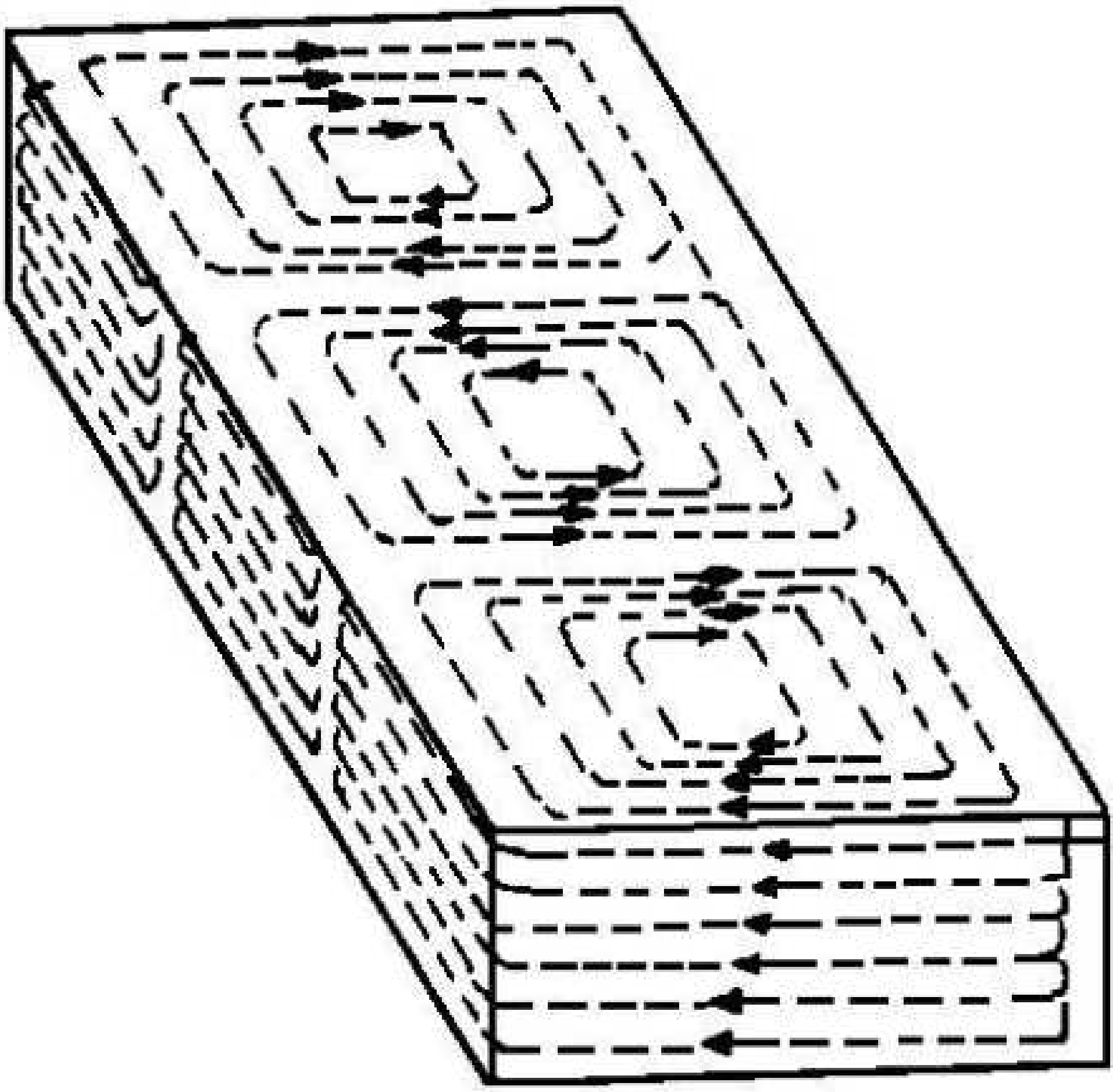

FIG. 5. E field distribution in a rectangular waveguide

2. A part of the program was taken from the website www.toyfdtd.com

3. Part of the algorithm and workings of FDTD was taken from the webpage of EPSRC Summer School 2007, Ian Drumm.

4. The mathematical analysis of the eigenmodes of propagation inside a waveguide was developed with help of the book Microwave engineering,David M.Pozar,Wiley publications,Third edition.